NOTES ON THE STRUCTURE OF SOLUTION SPACES

Throughout this review note, we consider the following cases

a. $T : V \to W$ is a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W.

b. A linear system $Ax = b$, where A is an $m \times n$ matrix and b is an m-column vector.

c. An n-th order differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots a_n(x)y = F(x). \tag{1}$$

Here $a_1(x), \cdots, a_n(x), F(x)$ are continuous.

d. A first order linear system of differential equation

$$X'(t) = A(t)X(t) + b(t) \tag{2}$$

with $A(t)$ being an $n \times n$ matrix of functions and $b(t)$ being an n-column vector of function. Here $A(t), b(t)$ are continuous.

Part I. The structure of the solution space to the homogeneous case.

a. The solution space to $Tv = 0$ forms a subspace, $\mathrm{Ker}(T)$, of V.

b. The solution space to $Ax = 0$ forms a subspace, the null space of A, of $\mathbb{R}^n$.

c. The solution space to

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots a_n(x)y = 0 \tag{3}$$

forms a **n**-dimension subspace of $C^n(I)$.

d. The solution space to

$$X'(t) = A(t)X(t) \tag{4}$$

forms a **n**-dimension subspace of $C^1_v(I)$, the space of n-times continuously differentiable vector functions.

Part II. The structure of the solution space to the inhomogeneous case.

a. The solution space to

$$Tv = w, \quad w \neq 0_W$$

is of the form

$$v_p + \mathrm{Ker}(T),$$

where v_p is a particular solution to $Tv = w$. Note this is not a vector space for $w \neq 0_W$.

b. The solution space to

$$Ax = b, \quad b \neq \vec{0}$$

is of the form

$$x_p + N(A),$$

where x_p is a particular solution to $Ax = b$ and $N(A)$ is the Null space of A. Note this is not a vector space for $b \neq \vec{0}$.

c. The solution to

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots a_n(x)y = F(x), \quad F \not\equiv 0 \tag{5}$$

is of the form

$$y_p + y_c,$$

where y_p is a particular solution to (5) and y_c is the general solution to (3). Note this solution space is not a vector space for $F \not\equiv 0$.

d. The solution to

$$X'(t) = A(t)X(t) + b(t), \quad b \not\equiv \vec{0} \tag{6}$$

is of the form

$$X_p + X_c,$$

where X_p is a particular solution to (6) and X_c is the general solution to (4). Note this solution space is not a vector space for $b \not\equiv \vec{0}$.

Remark 0.1. *We can view (b)-(d) from a linear transformation point of view as follows.*

b. $A : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation.*

$$\dim(N(A)) = n - \dim(\mathrm{Rng}(A)), \quad \dim(\mathrm{Rng}(A)) = \dim(\mathrm{colspace}(A)) = \mathrm{rank}(A)$$

c. *The map* $\mathcal{L} : C^n(I) \to C(I) : [y \mapsto a_1(x)y^{(n-1)} + \cdots a_n(x)y]$ *is a linear transformation.*

$$\dim(\mathrm{Ker}(\mathcal{L})) = n, \quad \dim(\mathrm{Rng}(\mathcal{L})) = \infty.$$

Indeed, by the existence and uniqueness theorem, for any $F(X)$ *in* $C(I)$*, there exists a solution* y *in* $C^n(I)$ *to* (1)*, i.e.,* $\mathcal{L}y = F$*. This implies*

$$\mathrm{Rng}(\mathcal{L}) = C(I).$$

d. *The map* $T : C_v^1(I) \to C_v(I) : [X \mapsto \frac{d}{dt}X - A(t)X]$ *is a linear transformation.*

$$\dim(\mathrm{Ker}(T)) = n, \quad \dim(\mathrm{Rng}(T)) = \infty.$$

Indeed, by the existence and uniqueness theorem, for any $b(t)$ *in* $C_v(I)$*, there exists a solution* X *in* $C_v^1(I)$ *to* (2)*, i.e.,* $TX = b$*. This implies*

$$\operatorname{Rng}(T) = C_v(I).$$

From the above point of view (b)-(d) in Part I and II are just special cases of (a). (Ignoring the fact that the liner map sin (c) and (d) are between infinitely dimensional spaces.)

Note that throughout this note, we require neither the coefficients $a_1(x), \cdots, a_n(x)$ *in (b) nor the coefficient matrix* $A(t)$ *in (c) to be constants.*